14 Perpendicularity and Angle Congruence

Definition (acute angle, right angle, obtuse angle, supplementary angles, complementary angles)
An acute angle is an angle whose measure is less than 90. A right angle is an angle whose measure is 90. An obtuse angle is an angle whose measure is greater than 90. Two angles are supplementary if the sum of their measures is 180. Two angles are complementary if the sum of their measures is 90.

Definition (linear pair of angles, vertical pair of angles)
Two angles $\measuredangle ABC$ and $\measuredangle CBD$ form a linear pair if $A-B-D$. Two angles $\measuredangle ABC$ and $\measuredangle A'BC'$ form a vertical pair if their union is a pair of intersecting lines. (Alternatively, $\measuredangle ABC$ and $\measuredangle A'BC'$ form a vertical pair if either $A-B-A'$ and $C-B-C'$, or $A-B-C'$ and $C-B-A'$.)

Theorem If C and D are points of a protractor geometry and are on the same side of $\overleftrightarrow{AB}$ and $m(\measuredangle ABC) < m(\measuredangle ABD)$, then $C \in \text{int}(\measuredangle ABD)$.

1. Prove the above theorem.

[Theorem 5.3.1, page 104]

Theorem (Linear Pair Theorem). If $\measuredangle ABC$ and $\measuredangle CBD$ form a linear pair in a protractor geometry then they are supplementary.

2. Prove the above theorem.

[Theorem 5.3.2, page 105]

3. If $A'-V-A$, $B'-V-B$, and $\measuredangle AVB$ is a right angle, then each of $\measuredangle AVB'$, $\measuredangle A'VB$, and $\measuredangle A'VB'$ is a right angle.

Theorem In a protractor geometry, if $m(\measuredangle ABC) + m(\measuredangle CBD) = m(\measuredangle ABD)$, then $C \in \text{int}(\measuredangle ABD)$.

4. Prove the above theorem.

[Theorem 5.3.3, page 106]

Note that the result about distance that corresponds to above Theorem is false. If $AB + BC = AC$ it need not be true that $B \in \text{int}(AB)$.

Theorem In a protractor geometry, if A and D lie on opposite sides of $\overleftrightarrow{BC}$ and if $m(\measuredangle ABC) + m(\measuredangle CBD) = 180$, then $A-B-D$ and the angles form a linear pair.

5. Prove the above theorem.

Definition (perpendicular lines, perpendicular rays, perpendicular segments)
Two lines ℓ and ℓ' are perpendicular (written $\ell \perp \ell'$) if $\ell \cup \ell'$ contains a right angle. Two rays or segments are perpendicular if the lines they determine are perpendicular.

6. If a is a segment, ray, or line and b is a segment, ray, or line, then $a \perp b$ implies $b \perp a$.

Theorem If P is a point on line ℓ in a protractor geometry, then there exists a unique line through P that is perpendicular to ℓ.

7. Prove the above theorem.

8. In the Poincaré Plane, find the line through $B(3,4)$ that is perpendicular to the line ${}_0L_5 = \{(x,y) \in \mathbb{H} \mid x^2 + y^2 = 25\}$.

[Example 5.3.6, page 107]

Corollary In a protractor geometry, every line segment $\overline{AB}$ has a unique perpendicular bisector; that is, a line $\ell \perp \overline{AB}$ with $\ell \cap \overline{AB} = \{M\}$ where M is the midpoint of $\overline{AB}$.

9. Prove the above corollary.

Theorem In a protractor geometry, every angle $\measuredangle ABC$ has a unique angle bisector that is, a ray $\overrightarrow{BD}$ with $D \in \text{int}(\measuredangle ABC)$ and $m(\measuredangle ABD) = m(\measuredangle DBC)$.

10. Prove the above theorem.

Definition (angle congruence)
In a protractor geometry $\{\mathcal{S}, \mathcal{L}, d, m\}$, $\measuredangle ABC$ is congruent to $\measuredangle DEF$ (written as $\measuredangle ABC \cong \measuredangle DEF$ if $m(\measuredangle ABC) = m(\measuredangle DEF)$.

11. Congruence of angles is an equivalence relation on the set of all angles.

12. Prove that any two right angles in a protractor geometry are congruent.

Theorem (Vertical Angle Theorem). In a protractor geometry, if $\measuredangle ABC$ and $\measuredangle A'BC'$ form a vertical pair then $\measuredangle ABC \cong \measuredangle A'BC'$.

13. Prove the above theorem.

Theorem (Angle Construction Theorem). In a protractor geometry, given $\measuredangle ABC$ and a ray $\overrightarrow{ED}$ which lies in the edge of a half plane H_1, then there exists a unique ray $\overrightarrow{EF}$ with $F \in H_1$ and $\measuredangle ABC \cong \measuredangle DEF$.

14. Prove the above theorem.

Theorem (Angle Addition Theorem). In a protractor geometry, if $D \in \operatorname{int}(\measuredangle ABC)$, $S \in \operatorname{int}(\measuredangle PQR)$, $\measuredangle ABD \cong \measuredangle PQS$, and $\measuredangle DBC \cong \measuredangle SQR$, then $\measuredangle ABC \cong \measuredangle PQR$.

15. Prove the above theorem.

Theorem (Angle Subtraction Theorem). In a protractor geometry, if $D \in \operatorname{int}(\measuredangle ABC)$, $S \in \operatorname{int}(\measuredangle PQR)$, $\measuredangle ABD \cong \measuredangle PQS$, and $\measuredangle ABC \cong \measuredangle PQR$, then $\measuredangle DBC \cong \measuredangle SQR$.

16. Prove the above theorem.

17. Show that if $\triangle ABC$ is in Poincaré plane with $A(0,1)$, $B(0,5)$, and $C(3,4)$ (this triangle we had earlier), then $(AC)^2 \neq (AB)^2 + (BC)$. Thus the Pythagorean Theorem need not be true in a protractor geometry.

18. In $\mathcal{H}$ find the angle bisector of $\measuredangle ABC$ if $A = (0,5)$, $B = (0,3)$ and $C = (2, \sqrt{21})$.

19. Prove that in a protractor geometry $\measuredangle ABC$ is a right angle if and only if there exists a point D with $D-B-C$ and $\measuredangle ABC \cong \measuredangle ABD$.

20. In the Taxicab Plane let $A = (0,2)$, $B = (0,0)$, $C = (2,0)$, $Q = (-2,1)$, $R = (-1,0)$ and $S = (0,1)$. Show that $\overline{AB} \cong \overline{QR}$, $\measuredangle ABC \cong \measuredangle QRS$, and $\overline{BC} \cong \overline{RS}$. Is $\overline{AC} \cong \overline{QS}$?

15 Euclidean and Poincaré Angle Measure

In this optional section we shall carefully verify that the Euclidean and Poincaré angle measures defined in Section 13 actually satisfy the axioms of an angle measure. The key step will be the construction of an inverse cosine function. This will involve techniques quite different from those of the rest of this course. As a result, you may choose to omit this section knowing that the only results that we will use in the sequel are that m_E and m_T are angle measures and that the cosine function is injective. On the other hand, it is interesting to see a variety of mathematical techniques tied together to develop one concept as is done in this section. The material on the construction of Euclidean angle measure is taken from Parker [1980].

Precisely what are we assuming in this section? We are assuming the standard facts about differentiation and integration but nothing about trigonometric functions. This will force us to consider the notion of an improper integral in order to define the inverse cosine function. Since general results about differential equations may not be familiar to the reader, we shall need to develop some very specific theorems regarding the solutions of $y'' = -y$. (In calculus we learned that both $sin(x)$ and $cos(x)$ are solutions of this differential equation. That is why we are interested in this equation.)

Definition (improper integral)
Let $f(t)$ be a function which is continuous for $c \le t < d$ and which may not be defined at $t = d$. Then the improper integral $\int_c^d f(t)\,\mathrm{d}t$ converges if $\lim_{b\to d^-} f(t)\,\mathrm{d}t$ exists. In this case, we say $\lim_{b\to d^-} f(t)\,\mathrm{d}t = \int_c^d f(t)\,\mathrm{d}t$.

Lemma The improper integral $\displaystyle\int_0^1 \frac{\mathrm{d}t}{\sqrt{1-t^2}}$ converges. [Lemma 5.4.1, page 110]

A similar argument shows that the improper integral $\int_{-1}^0 \frac{\mathrm{d}t}{\sqrt{1-t^2}}$ converges so that $\int_0^{-1} \frac{\mathrm{d}t}{\sqrt{1-t^2}}$ also exists. We define a number p to be twice the value of the integral in above lemma: ...

...(see book, pages 109-123)...

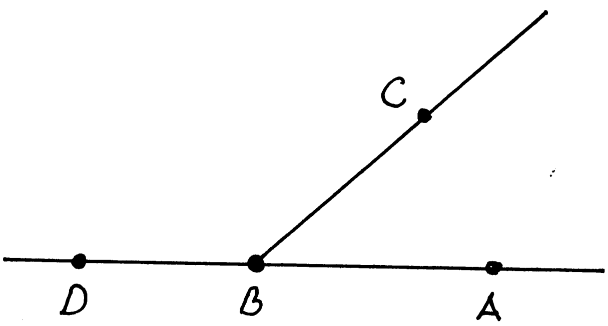
Okomitost i podudarnost uglova

Definicija (oštri ugao, pravi ugao, tupi ugao, suplementarni uglovi, komplementarni uglovi)

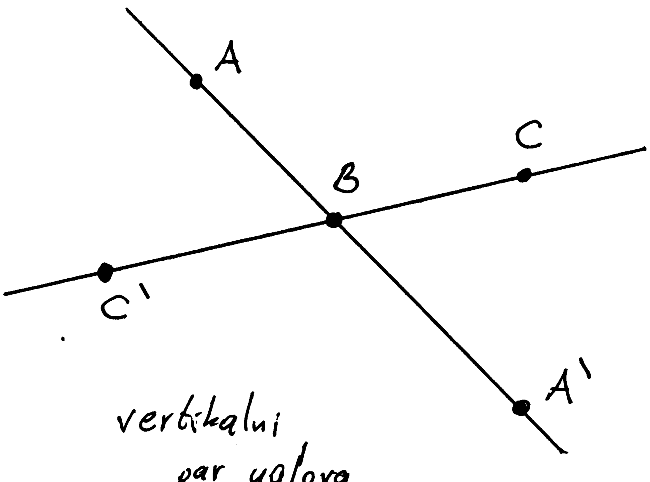
Oštri ugao je ugao čija je mjera manja od 90. Pravi ugao je ugao čija je mjera jednaka 90. Tupi ugao je ugao čija je mjera veća od 90. Dva ugla su suplementarna ako suma njihovih mjera iznosi 180. Dva ugla su komplementarna ako je suma njihovih mjera 90.

Definicija (linearni par uglova, vertikalni par uglova)

Dva ugla $\measuredangle ABC$ i $\measuredangle CBD$ formiraju linearan par ako je $A-B-D$. Dva ugla $\measuredangle ABC$ i $\measuredangle A'BC'$ formiraju vertikalan par ako je njihova unija par pravih koje se sijeku.



linearni par uglova



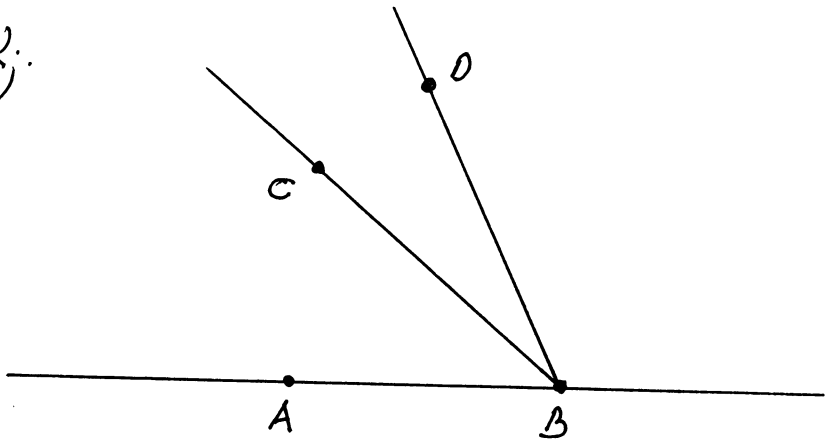
vertikalni par uglova

Teorema

Ako su C i D tačke protractor geometrije, ako su sa iste strane prave $p(A,B)=\overleftrightarrow{AB}$ i ako je $m(\measuredangle ABC) < m(\measuredangle ABD)$ tada je $C \notin int(\measuredangle ABD)$.

Dokazati teoremu iznad.

Rj.



Želimo upotrebiti sljed. teor.

Teor.

$P \in int(\measuredangle ABC)$ akko
 A i P sa iste str. $\overleftrightarrow{BC}$ i
 C i P sa iste str. pr. $\overleftrightarrow{BA}$

Skica dokaza:

Za tačke A i C moguć je jedan od sljedeća tri slučaja:

1° A i C su sa iste strane $p(B,D)=\overleftrightarrow{BD}$

2° $C \in \overleftrightarrow{BD}$

3° A i C su sa različitih strana prave $\overleftrightarrow{BD}$.

Pokazaćemo da slučajevi 2 i 3 nisu moguća.

$C \in \overleftrightarrow{BD}$, C i D sa iste str. $\overleftrightarrow{AB}$ $\Rightarrow$ $C \in int(\overrightarrow{BD})$ $\Rightarrow$

$\Rightarrow$ $\measuredangle ABC = \measuredangle ABD$ i $m(\measuredangle ABC) = m(\measuredangle ABD)$ #kontradikcija

A i C su razl. str. $\overleftrightarrow{BD}$ $\xRightarrow{\text{jednom od ranijih probl.}}$ $D \in int(\measuredangle ABC)$ $\Rightarrow$

$\Rightarrow$ $m(\measuredangle ABD) + m(\measuredangle DBC) = m(\measuredangle ABC) < m(\measuredangle ABD)$ $\Rightarrow$

$\Rightarrow$ $m(\measuredangle DBC) < 0$. #kontradikcija

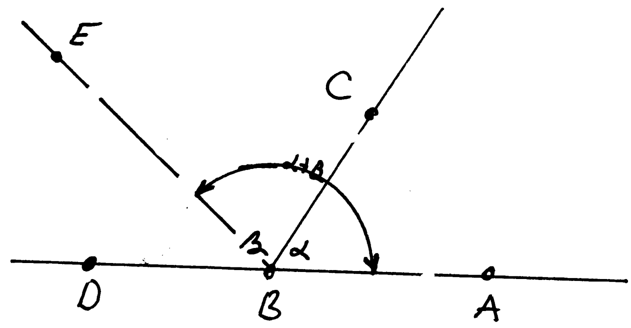
Pa jedina mogućnost je da su A i C sa iste strane prave $\overleftrightarrow{BD}$

$\Rightarrow$ $C \in int(\measuredangle ABD)$.

Teorema (teorem linearnog para)

Ako uglovi $\measuredangle ABC$ i $\measuredangle CBD$ formiraju linearni par u protractor geometriji tada su oni suplementarni.

(#) Dokazati teoremu iznad.



Rj:

Skica dokaza:

$m(\measuredangle ABC)=\alpha$

$m(\measuredangle CBD)=\beta$

Pok. $\alpha+\beta=180$ (pokazaćemo da $\alpha+\beta<180$ i $\alpha+\beta>180$ vodi u kontradikciju)

Pretp. $\alpha+\beta<180$.

Prema aksiomu konstrukcije ugla $\exists!\ \overrightarrow{BE}$ sa tačk. E na istoj str. pr. $\overleftrightarrow{AB}$ kao C

$$m(\measuredangle ABE)=\alpha+\beta$$

Preth. teor $\Rightarrow$ $C\in int(\measuredangle ABE)$ $\Rightarrow$ $m(\measuredangle ABC)+m(\measuredangle CBE)=m(\measuredangle ABE)$

Time

$\alpha+m(\measuredangle CBE)=\alpha+\beta$ $\Rightarrow$ $m(\measuredangle CBE)=\beta$

E i C su sa iste strane prave $\overleftrightarrow{AD}$
$m(\measuredangle ABC)<m(\measuredangle ABE)$
i vrijedi $A-B-D$

Sa druge str. $E\in int(\measuredangle CBD)$ (ZAŠTO?) pa je

$$m(\measuredangle CBE)+m(\measuredangle EBD)=m(\measuredangle CBD) \Rightarrow$$

$$\Rightarrow \beta+m(\measuredangle EBD)=\beta \Rightarrow m(\measuredangle EBD)=0$$

#kontradikcija.
(mjera ugla je uvijek veća od nule)

Prema tome slučaj $\alpha+\beta<180$ nije moguć.

Sad pretpostavimo da je $\alpha+\beta>180$.

(nastavak dokaza pogledaj u knjizi, Teorema 5.3.2)

$\left.\begin{array}{l}\alpha<180\\ \beta<180\end{array}\right\} \Rightarrow \alpha+\beta<360 \Rightarrow 0<\alpha+\beta-180<180$ $\gamma=\alpha+\beta-180 \xRightarrow{\beta<180} \gamma<\alpha$

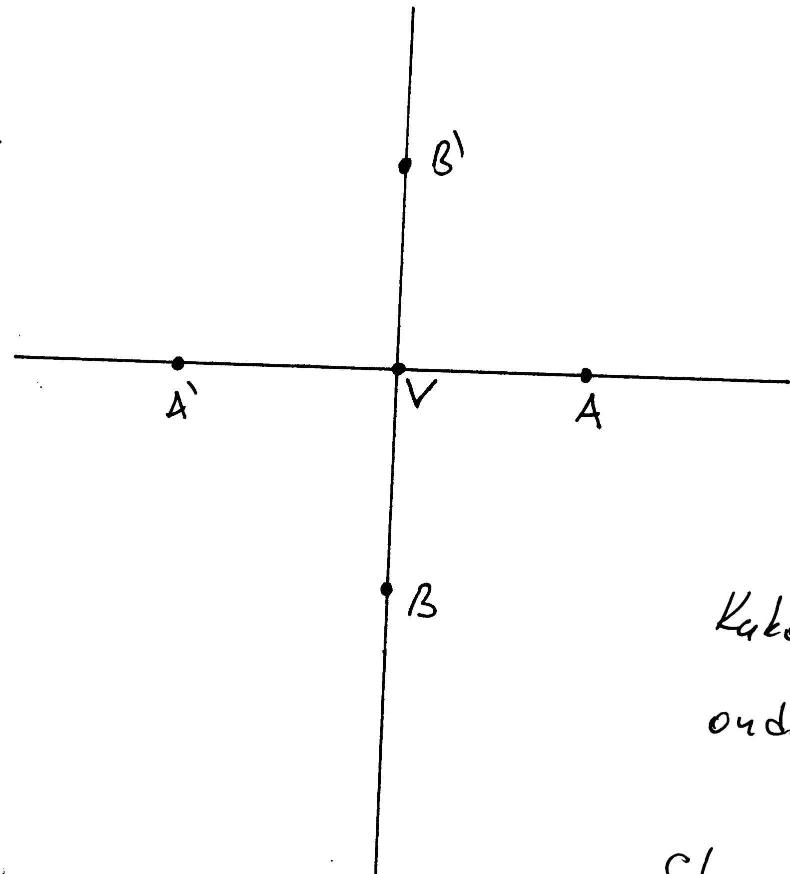
Neka je H poluravan sa ivicom u $\overleftrightarrow{AD}$ koja sadrži tačku C.

Prema Protractor postulatu $\exists!\ \overrightarrow{BF}$ t.d. $F\in H$ i $m(\measuredangle ABF)=\gamma$

C i F su sa iste strane prave $\overleftrightarrow{AD}$ i $m(\measuredangle ABF)<m(\measuredangle ABC)$ $\Rightarrow$ $F\in int(\measuredangle ABC)$

(#) Ako je $A'-V-A$, $B'-V-B$ i $\measuredangle AVB$ je pravi ugao, pokazati da je tada svaki od uglova $\measuredangle AVB'$, $\measuredangle A'VB$ i $\measuredangle A'VB'$ pravi ugao.

Rj. $\measuredangle AVB$ prav ugao $\Rightarrow$ $m(\measuredangle AVB) = 90$.



Primjetimo da $\measuredangle AVB$ i $\measuredangle AVB'$ formira linearan par uglova, pa su oni suplementarni, što znači da je

$$m(\measuredangle BVA) + m(\measuredangle AVB') = 180$$

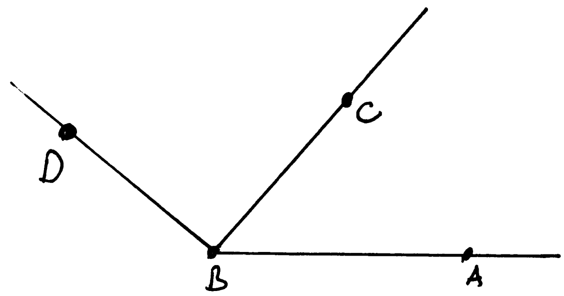
vidi prethodnu teoremu
(vidi jednu od prethodnih teorema).

Kako je $m(\measuredangle BVA) = 90$ to je onda $m(\measuredangle AVB') = 90$ $\Rightarrow$ $\measuredangle AVB'$ je prav ugao.

Slično za druga dva ugla.

Teorema

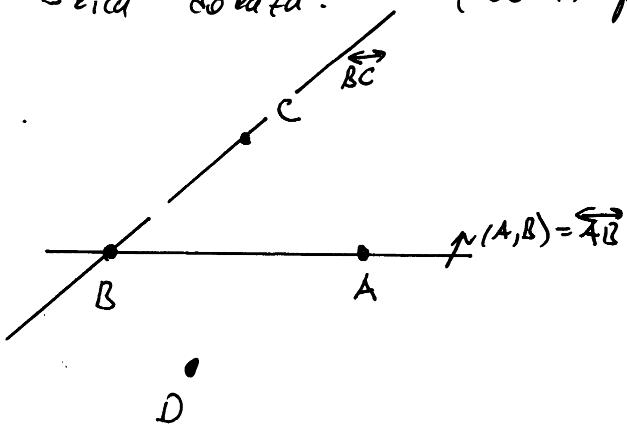
U protractor geometriji, ako je $m(\sphericalangle ABC)+m(\sphericalangle CBD)=m(\sphericalangle ABD)$ tada je $C\in int(\sphericalangle ABD)$.



(#) Dokazati teoremu iznad.

Rj.

Skica dokaza: (dokaz potraži u knjizi, Teorem 5.3.3)



Kontradikcijom ćemo pok. da C i D pripadaju istoj strani prave $\overleftrightarrow{AB}$.
Pretpostavimo da C i D pripadaju različ. stranama prave $\overleftrightarrow{AB}$

$A\notin\overleftrightarrow{BC}$, $D\notin\overleftrightarrow{BC}$

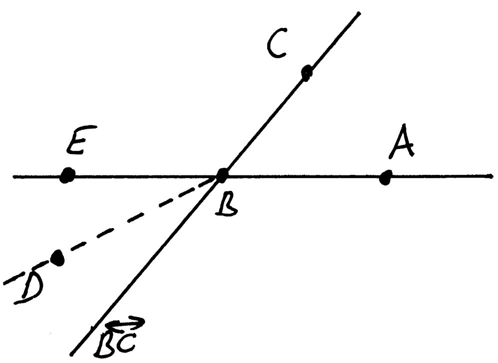
Ako A i D pripadaju istoj strani prave $\overleftrightarrow{BC}$ $\Rightarrow$ $A\in int(\sphericalangle CBD)$

ZAŠTO?

$\Rightarrow$ $m(\sphericalangle CBA)+m(\sphericalangle ABD)=m(\sphericalangle CBD)<m(\sphericalangle ABD)$

#kontradikcija

Prema tome, A i D su sa različitih strana $\overleftrightarrow{BC}$.



Izaberimo E t.d. $A-B-E$ i primjetimo da su E i D sa iste strane prave $\overleftrightarrow{BC}$

$\Rightarrow$ $E\in int(\sphericalangle CBD)$ (ZAŠTO?) $\Rightarrow$

$$m(\sphericalangle CBE)+m(\sphericalangle EBD)=m(\sphericalangle CBD)$$

Kako $\sphericalangle ABC$ i $\sphericalangle CBE$ formiraju linearan par, $m(\sphericalangle CBE)=180-m(\sphericalangle ABC)$

$\Rightarrow$ $180-m(\sphericalangle ABC)+m(\sphericalangle EBD)=m(\sphericalangle CBD)$ $\Rightarrow$ $180+m(\sphericalangle EBD)=\underbrace{m(\sphericalangle ABC)+m(\sphericalangle CBD)}_{=m(\sphericalangle ABD)}$

$\Rightarrow$ $m(\sphericalangle ABD)>180$

#kontradikcija

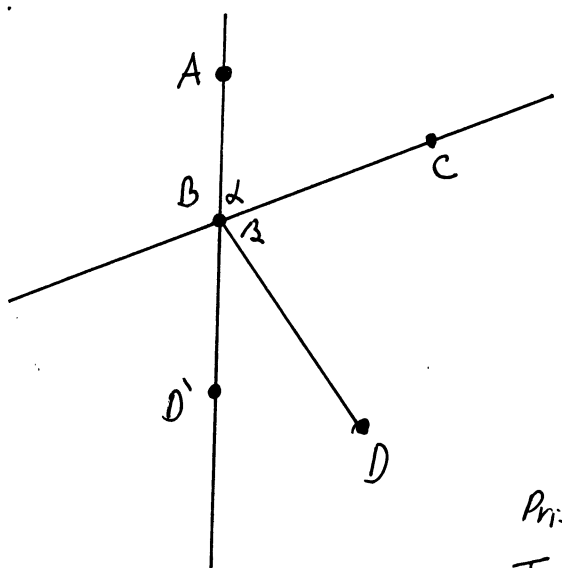
Prema tome C i D ne mogu biti sa različitih strana prave $\overleftrightarrow{AB}$. Sad upotrebi Teor. 5.3.1

Teorema

U protractor geometriji, ako A i D pripadaju različitim stranama prave $\overleftrightarrow{BC}$ i ako je $m(\measuredangle ABC)+m(\measuredangle CBD)=180$ tada $A-B-D$ i uglovi formiraju linearan par.

\# Dokazati teoremu iznad.

Rj.



Uvedimo oznake $\alpha = m(\measuredangle ABC)$

$\beta = m(\measuredangle CBD)$

$\alpha+\beta=180$

Neka je $D' \in \overleftrightarrow{AB}$, $A-B-D'$

Primjetimo da uglovi $\measuredangle ABC$ i $\measuredangle CBD'$ formiraju linearan par.

Prisjetimo se

Teor. Ako $\measuredangle ABC$ i $\measuredangle CBD$ form. lin. par. $\Rightarrow$ $\measuredangle ABC$ i $\measuredangle CBD$ su suplem. uglovi ...(1)

Na osnovu (1) imam da $m(\measuredangle ABC)+m(\measuredangle CBD')=180°$ $\Rightarrow$

$\Rightarrow$ $\alpha + m(\measuredangle CBD') = \alpha+\beta$ $\Rightarrow$ $m(\measuredangle CBD')=\beta$

Primjetimo da D i D' pripadaju istoj strani prave $\overleftrightarrow{BC}$.
S obzirom da je $m(\measuredangle CBD)=\beta=m(\measuredangle CBD')$ to slučajevi da je $D\in \operatorname{int}(\measuredangle CBD')$ i $D'\in \operatorname{int}(\measuredangle CBD)$ nisu moguća $\Rightarrow$ $D \in \operatorname{int}(\overrightarrow{BD'})$

$\Rightarrow$ $A-B-D$

i uglovi $\measuredangle ABC$ i $\measuredangle CBD$ formiraju linearni par.

Definicija (okomitost)

Dvije prave l i l' su okomite (ili normalne) (što označavamo sa $l \perp l'$) ako $l \cup l'$ sadrži prav ugao.

Dvije poluprave ili dva segmenta su okomita ako su okomite prave koje one određuju.

\# Ako je a duž (poluprava ili prava) i ako je b duž (poluprava ili prava) tada $a \perp b$ povlači da $b \perp a$.

Rj.

Pretpostavimo da je a na pravoj $\overleftrightarrow{AB}$, a da je b na pravoj $\overleftrightarrow{CD}$. Tada

$a \perp b$ akko $\overleftrightarrow{AB} \perp \overleftrightarrow{CD}$ akko $\overleftrightarrow{AB} \cup \overleftrightarrow{CD}$ sadrži prav ugao akko $\overleftrightarrow{CD} \cup \overleftrightarrow{AB}$ sadrži prav ugao akko $\overleftrightarrow{CD} \perp \overleftrightarrow{AB}$ akko $b \perp a$.

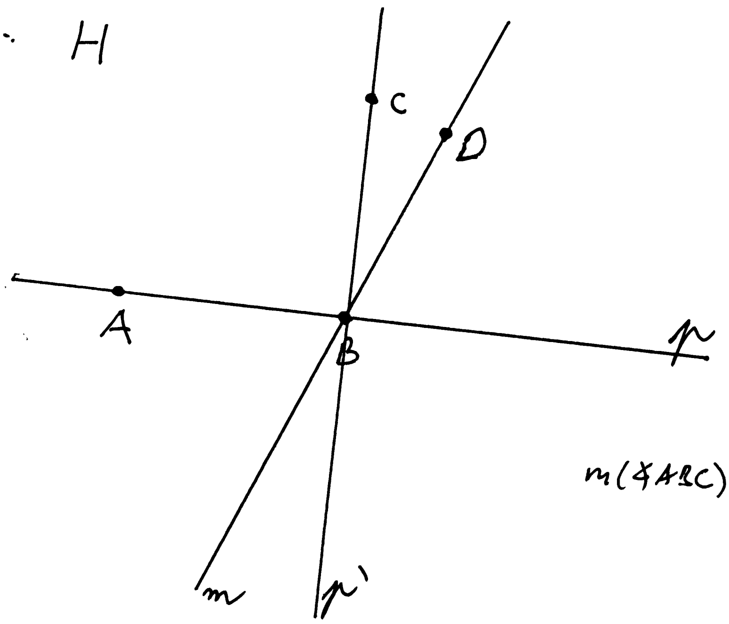
Prema tome $a \perp b \Leftrightarrow b \perp a$.

Teorem

Za datu pravu p i tačku $B \in p$ u protractor geometriji, postoji jedinstvena prava p' koja sadrži tačku B i ima osobinu da $p \perp p'$.

(#) Dokazati teoremu iznad.

Rj.:



Neka je $p = p(AB) = \overleftrightarrow{AB}$.
Označimo sa H poluravan sa ivicom u p. Prema osobini (ii) Protractor postulata postoji jedinstvena poluprava $\overrightarrow{BC}$ t.d. $C \in H$ i $m(\measuredangle ABC) = 90$.

$m(\measuredangle ABC) = 90^\circ \Rightarrow \measuredangle ABC$ je prav ugao.

Prema tome $\overleftrightarrow{BC}$ je prava kroz tačku B koja je okomita na p. Drugim rečima $p' = \overleftrightarrow{BC}$.
($p \cup p'$ sadrže prav ugao $\Rightarrow p \perp p'$)

Pokažimo jedinstvenost prave.

Neka je m prava kroz tačku B koja je okomita na pravu p.

Kako $m \cap p = \{B\}$ i $m \neq p \Rightarrow \exists D$ t.d. $D \in m$, $D \in H$.

$m \perp p \Rightarrow \measuredangle ABD$ je pravi ugao (vidi jedan od prethodnih zadataka)

$\Rightarrow m(\measuredangle ABD) = 90$.

Zbog jedinstvenosti prave $\overrightarrow{BC}$, imamo da je $\overrightarrow{BC} = \overrightarrow{BD}$. Prema tome $m = \overleftrightarrow{BC}$, i $\overleftrightarrow{BC}$ je jedinstvena prava kroz tačku B koja je okomita na p.

(#) U Poincaré-ovoj ravni, pronaći pravu kroz tačku $B(3,4)$ koja je okomita na pravu

$$_0L_5 = \{(x,y)\in \mathbb{H} \mid x^2+y^2=25\}.$$

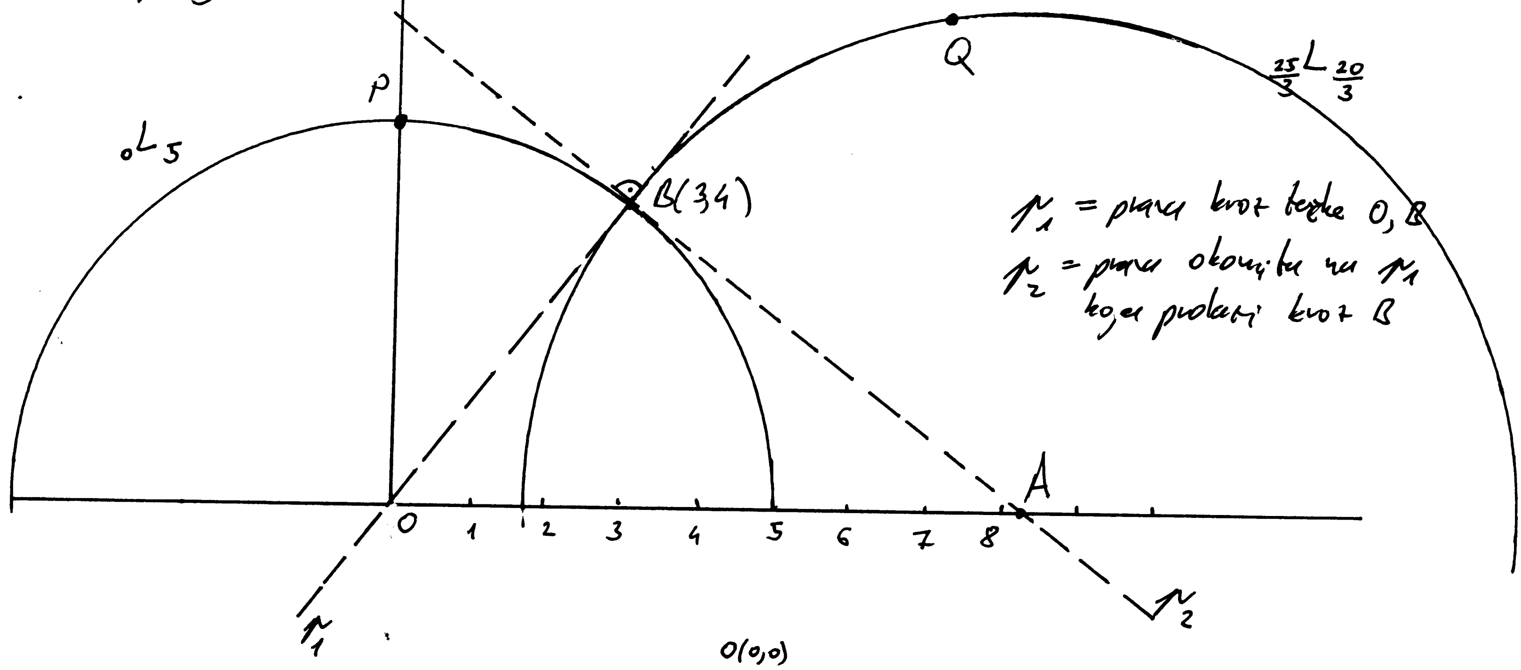
Rj.

$x^2+y^2=25$

$C(0,0)$

$r=5$

Nije teško iskoristiti znanje analitičke geometrije i dobiti rješenje.



$p_1 = p(0,B) = \overleftrightarrow{OB}$ $\quad \dfrac{x-x_1}{x_2-x_1} = \dfrac{y-y_1}{y_2-y_1} \overset{\substack{O(0,0)\\B(3,4)}}{\Longrightarrow} \dfrac{x}{3} = \dfrac{y}{4} \Rightarrow y = \frac{4}{3}x$

$p_1 : y = \frac{4}{3}x \Rightarrow k_2 = -\frac{3}{4}$ (k_2 je koeficijent pravca prave p_2)

$\left.\begin{array}{l} p_2 : y = k_2 x + n \\ B(3,4) \end{array}\right\} \Rightarrow 4 = -\frac{3}{4}\cdot 3 + n \Rightarrow 16 = -9 + 4n$

$4n = 25$

$n = \frac{25}{4}$

$p_2 : y = -\frac{3}{4}x + \frac{25}{4} \overset{y=0}{\Longrightarrow} \frac{3}{4}x = \frac{25}{4} \Rightarrow x = \frac{25}{3}$

Centar prave koje tražimo je $c = \frac{25}{3}$

$A(\frac{25}{3}, 0)$

$B(3,4)$

$AB = \sqrt{(3-\frac{25}{3})^2 + (4-0)^2} = \sqrt{\frac{16^2}{9} + 16} = \sqrt{\frac{16^2+16\cdot 9}{9}} = \frac{20}{3}$

Time smo dobili da $_{\frac{25}{3}}L_{\frac{20}{3}}$ vjerovatno zadovoljava date osobine. Ovo sad trebamo provjeriti koristeći formule iz prethodne lekcije.

Pa da bi provjerili da li je dobijeno rješenje dobro, pored tačke B trebaju nam tačke P i Q t.d. $P \in {}_0L_5$, $Q \in {}_{\frac{25}{3}}L_{\frac{20}{3}}$.

${}_0L_5: \quad x^2 + y^2 = 25$

$x = 0 \Rightarrow y = 5$

$P(0, 5)$

${}_{\frac{25}{3}}L_{\frac{20}{3}}: \quad \left(x - \frac{25}{3}\right)^2 + y^2 = \frac{20}{3}$

$x = \frac{22}{3} \qquad \frac{22}{3} - \frac{25}{3} = \frac{22-25}{3} = \frac{-3}{3} = -1$

$\Downarrow$

$y = \sqrt{\frac{20}{3} - 1} = \sqrt{\frac{17}{3}}$

$Q\left(\frac{22}{3}, \sqrt{\frac{17}{3}}\right)$

$x = \frac{19}{3}$

$\Downarrow$

$y = \sqrt{\frac{8}{3}}$

Sad trebamo izračunati mjeru ugla $\measuredangle PBQ$.
Za tu svrhu nam prvo treba T_{BP} i T_{BQ} ...

Corolar

U protractor geometriji svaka duž $\overline{AB}$ ima jedinstvenu simetralu; tj. pravu $s \perp \overline{AB}$ taka da $s \cap \overline{AB} = \{M\}$ gdje je M sredina duži $\overline{AB}$.

Dokazati Corolar iznad.

Rj.

Od ranije znamo da svaka duž $\overline{AB}$ ima sredinu M, i da je tačka M jedinstvena. Prisjetimo se ukratko dokaza ove tvrdnje.

Neka je $f: \overleftrightarrow{AB} \to \mathbb{R}$ koordinatni sistem takav da je $f(A)=0$, $f(B)=b$

Definišimo $M \in \overleftrightarrow{AB}$ sa $M = f^{-1}\left(\frac{b}{2}\right)$. Kako je f surjekcija (na $\mathbb{R}$), M postoji, i kako je f 1-1 M je jedinstveno određena.

$$\left.\begin{aligned} d(A,M) &= |f(A) - f(M)| = \frac{b}{2} \\ d(M,B) &= |f(M) - f(b)| = \frac{b}{2} \end{aligned}\right\} \Rightarrow AM = MB \text{ i } M \text{ je jedinstvena tačka duži } \overline{AB}$$

Pa ako je data duž $\overline{AB}$, neka je M sredina duži.

Prema prethodnom teoremu: Za datu pravu ℓ i tačku $B \in \ell$ u protrac. geom., postoji jedinstv. prava ℓ' t.d. $B \in \ell'$ i $\ell \perp \ell'$.

Sad kako je $M \in \overleftrightarrow{AB}$ to prema teoremi iznad postoji jedinstvena prava s t.d. $M \in s$ i $s \perp \overleftrightarrow{AB}$.

Tada je i $s \perp \overline{AB}$ i $s \cap \overline{AB} = \{M\}$, gdje je M sredina duži $\overline{AB}$.

Definicija

U protractor geometriji $\{\mathcal{S}, \mathcal{L}, d, m\}$, ugao $\measuredangle ABC$ je podudaran sa uglom $\measuredangle DEF$ (ovo označavamo sa $\measuredangle ABC \cong \measuredangle DEF$) ako je

$$m(\measuredangle ABC) = m(\measuredangle DEF)$$

(#) Pokazati da su svaka dva prava ugla u protractor geometriji jednaka.

Rj.: Neka su $\measuredangle ABC$ i $\measuredangle PQR$ proizvoljna dva prava ugla.

$$\left.\begin{array}{l} \measuredangle ABC \text{ prav} \Rightarrow m(\measuredangle ABC) = 90 \\ \measuredangle PQR \text{ prav} \Rightarrow m(\measuredangle PQR) = 90 \end{array}\right\} \Rightarrow m(\measuredangle ABC) = m(\measuredangle PQR)$$

$$\Downarrow$$

$$\measuredangle ABC \cong \measuredangle PQR.$$

(#) Kongruencija uglova je relacija ekvivalencije na skupu svih uglova.

Rj.

REFLEKSIVNOST

Kako je $m(\measuredangle ABC) = m(\measuredangle ABC)$ to je $\measuredangle ABC \cong \measuredangle ABC$

SIMETRIČNOST

$\measuredangle ABC \cong \measuredangle PQR \Rightarrow m(\measuredangle ABC) = m(\measuredangle PQR) \Rightarrow$

$\Rightarrow m(\measuredangle PQR) = m(\measuredangle ABC) \Rightarrow \measuredangle PQR \cong \measuredangle ABC$

TRANZITIVNOST

$\measuredangle ABC \cong \measuredangle PQR$ i $\measuredangle PQR \cong \measuredangle DEF \Rightarrow$

$\Rightarrow m(\measuredangle ABC) = m(\measuredangle PQR)$ i $m(\measuredangle PQR) = m(\measuredangle DEF)$

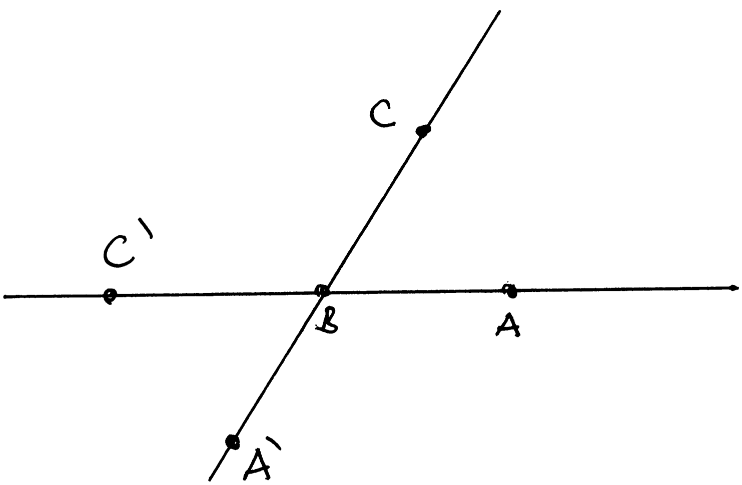
$\Rightarrow m(\measuredangle ABC) = m(\measuredangle DEF) \Rightarrow \measuredangle ABC \cong \measuredangle DEF$

Teorem (teorem vertikalnog ugla)

U protractor geometriji, ako $\measuredangle ABC$ i $\measuredangle A'BC'$ formira vertikalan par tada $\measuredangle ABC \cong \measuredangle A'BC'$.

\# Dokazati teoremu iznad.

Rj.



Pretpostavimo da je $A-B-C'$ i $A'-B-C$.

Primetimo da u tom slučaju uglovi $\measuredangle ABC$ i $\measuredangle CBC'$ formiraju linearan par. $\Rightarrow$

$$m(\measuredangle ABC) + m(\measuredangle CBC') = 180^\circ \quad \ldots (1)$$

S druge strane $\measuredangle(CBC')$ i $\measuredangle A'BC'$ form lin par.

$$m(\measuredangle CBC') + m(\measuredangle A'BC') = 180^\circ \quad \ldots (2)$$

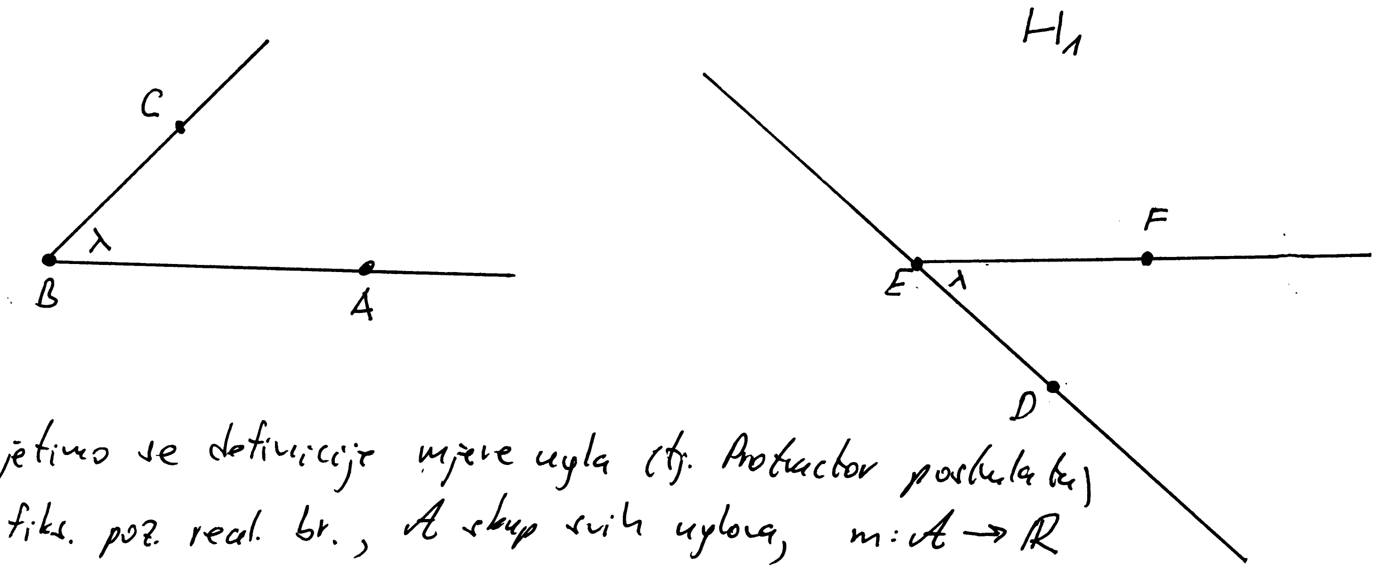
(1) i (2) $\Rightarrow$ $m(\measuredangle ABC) = m(\measuredangle A'BC')$ $\Rightarrow$ $\measuredangle ABC \cong \measuredangle A'BC'$.

Teorem (teorem konstrukcije ugla)

U protractor geometriji, za dati ugao $\measuredangle ABC$ i poluprava $\overrightarrow{ED}$ koja pripada ivici poluravni H_1, postoji jedinstvena poluprava $\overrightarrow{EF}$ taka da $F \in H_1$ i $\measuredangle ABC \cong \measuredangle DEF$.

Dokazati teoremu iznad.

Rj:



Prisjetimo se definicije mjere ugla (tj. Protractor postulata)
r_0 fiks. poz. real. br., $\mathcal{A}$ skup svih uglova, $m: \mathcal{A} \to \mathbb{R}$

(i) $\measuredangle ABC \in \mathcal{A} \Rightarrow 0 < m(\measuredangle ABC) < r_0$

(ii) $\overrightarrow{BC}$ ivica u poluravni H_1, i ako je θ real. broj:, $0 < \theta < r_0$
$\Rightarrow \exists$ jedinst. po. prava $pp[B,A) = \overrightarrow{BA}$ sa osob. $A \in H_1$, $m(\measuredangle ABC) = \theta$

(iii) $D \in int(\measuredangle ABC) \Rightarrow m(\measuredangle ABD) + m(\measuredangle DBC) = m(\measuredangle ABC)$.

Prema osobini (ii) protractor postulata $\exists!\ \overrightarrow{EF}$ sa osobinama

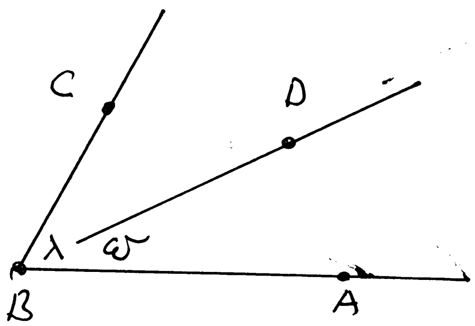
da $F \in H_1$ i $m(\measuredangle ABC) = m(\measuredangle DEF) \Rightarrow \measuredangle ABC \cong \measuredangle DEF$

Teorem (teorem za oduzimanje uglova)

U protractor geometriji, ako je $D \in \mathrm{int}(\measuredangle ABC)$, $S \in \mathrm{int}(\measuredangle PQR)$, $\measuredangle ABD \cong \measuredangle PQS$ i $\measuredangle ABC \cong \measuredangle PQR$ tada $\measuredangle DBC \cong \measuredangle SQR$.

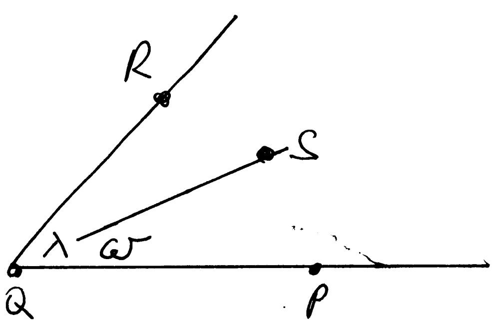
(#) Dokazati teoremu iznad.

Rj.



$$D \in \mathrm{int}(\measuredangle ABC) \xRightarrow{\text{prema osobini (iii) iz def. Protract. postulata}} m(\measuredangle ABD) + m(\measuredangle DBC) = m(\measuredangle ABC) \quad \ldots(1)$$

$$S \in \mathrm{int}(\measuredangle PQR) \xRightarrow{\text{prem osob. (iii) iz def. Protrac. post}} m(\measuredangle PQS) + m(\measuredangle SQR) = m(\measuredangle PQR) \quad \ldots(2)$$



$\measuredangle ABC \cong \measuredangle PQR \Rightarrow$

$\Rightarrow m(\measuredangle ABC) = m(\measuredangle PQR)$

označimo ove dvije mjere sa λ

tj. $m(\measuredangle ABC) = \lambda$
$m(\measuredangle PQR) = \lambda$

$\measuredangle ABD \cong \measuredangle PQS \Rightarrow m(\measuredangle ABD) = m(\measuredangle PQS)$

Označimo ove dvije mjere sa ω

tj. $m(\measuredangle ABD) = \omega$
$m(\measuredangle PQS) = \omega$

$$\left.\begin{array}{l} (1) \Rightarrow \omega + m(\measuredangle DBC) = \lambda \\ (2) \Rightarrow \omega + m(\measuredangle SQR) = \lambda \end{array}\right\} \Rightarrow m(\measuredangle DBC) = m(\measuredangle SQR)$$

$$\Downarrow$$

$$\measuredangle DBC \cong \measuredangle SQR$$

q.e.d.